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A Feigenbaum Sequence of Bifurcations in the Lorenz Model

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For some high values of the Rayleigh number r, the Lorenz model exhibits laminar behavior due to the presence of a stable periodic orbit. A detailed numerical study shows that, for r decreasing, the turbulent behavior is reached via an infinite sequence of bifurcations, whereas for r increasing, this is due to a collapse of the stable orbit to a hyperbolic one. The infinite sequence of bifurcations is found to be compatible with Feigenbaum's conjecture.

KEY WORDS: Lorenz equations; turbulence; strange attractors; periodic orbits; universal properties in sequences of infinite bifurcations.

1. INTRODUCTION

Lorenz[1] investigated the following remarkable system of three first-order differential equations, representing a flow in three-dimensional space:

$$\begin{aligned} \dot{x} &\quad -\sigma x + \sigma y \\ \dot{y} &= rx - y - xz \\ \dot{z} &= -bz + xy \end{aligned} \tag{1}$$

σ, b, and r being positive constants.

Numerical tests assuming $\sigma = 10$, $b = 8/3$, and r varying in certain ranges yield solutions heavily dependent on the initial condition and characterized by a chaotic, extremely random asymptotic behavior. Ruelle and Takens[2] explain this turbulent behavior as arising from the presence of an attractor of complex nature, which is termed a "strange attractor."

Recently, accurate studies on the Lorenz system have been reported by several authors, including Guckenheimer,[3] Marsden and McCracken,[4]

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Ruelle,[5] Williams,[6] and, particularly, Lanford,[7] who performed a detailed and exhaustive analysis of the strange attractor structure.

In Ref. 8 Henon studies the simple mapping of the plane

$$x_{i+1} = y_i + 1 - ax_i^2, \qquad y_{i+1} = bx_i$$

which has the same essential features as the Lorenz system. Actually, numerical tests for $a = 1.4$ and $b = 0.3$ show that, depending on the initial point (x_0, y_0), the sequence of points obtained by iterating the mapping either diverges to infinity or tends toward a strange attractor which appears to be a one-dimensional manifold times a Cantor set. The Henon model has been further examined by several workers, among them Feit[9] and Curry.[10]

So far the Lorenz and Henon models yield the best-known attempts at interpreting turbulence through the solution of evolution equations having a sensitive dependence on initial conditions.

In addition, a third model has been recently suggested by Boldrighini and Franceschini[11] and Franceschini and Tebaldi,[12] specified by the following equations:

$$\begin{aligned}
\dot{x}_1 &= -2x_1 + 4x_2x_3 + 4x_4x_5 \\
\dot{x}_2 &= -9x_2 + 3x_1x_3 \\
\dot{x}_3 &= -5x_3 - 7x_1x_2 + r \\
\dot{x}_4 &= -5x_4 - x_1x_5 \\
\dot{x}_5 &= -x_5 - 3x_1x_4
\end{aligned} \tag{2}$$

which have been obtained through a suitable five-mode truncation of the Navier–Stokes equations for a two-dimensional, incompressible fluid on a torus. In a certain range of the Reynolds number r, this system shows a turbulent behavior quite similar to that of the Lorenz model, which is due to the presence of two symmetrically placed strange attractors. Two distinctive features appear in this model:

(i) The presence of two (presumably) infinite sequences of orbits characterized by a period which doubles at each bifurcation. Moreover, these sequences are strictly related to the mechanism responsible for the generation of the strange attractor.

(ii) A mechanism which makes the turbulence disappear. One might think that this could arise because of the shrinking of the strange attractor to a periodic orbit, but it is in fact related to the simultaneous appearance of a stable orbit and a hyperbolic one on the same attractive manifold containing the strange attractor.

Concerning the pair of infinite sequences of bifurcations, a remarkable property has been verified, i.e., that they are compatible with a theory recently developed by Feigenbaum.[13] This theory is related to a large class of mappings of the interval onto itself, exhibiting an infinite sequence of bifurcations by varying a parameter λ. We briefly outline here the fundamental point of Feigenbaum's theory, while we refer to Ref. 13 for further details, and to Derrida *et al.*[14] and Collet *et al.*[15] for later generalizations of Feigenbaum's work. Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_n, \ldots$ be the bifurcation points of an infinite sequence of bifurcations; then (see the above references for the conditions under which this relation holds)

$$\lim_{i \to \infty} \frac{\Lambda_i - \Lambda_{i-1}}{\Lambda_{i+1} - \Lambda_i} = \delta = 4.6692 \cdots$$

Feigenbaum's conjecture and its extensions to higher dimensional systems also have been successfully checked within the framework of Henon's model too,[14] where the sequence of stable periods 2^n bifurcates with a velocity δ which is the same as that found in the case of the interval mappings.

In this paper we report some numerical results obtained on the Lorenz model. The aim of this work is to indicate that the same phenomena characterizing the system (2) are present in this model too. We exhibit an interval $[R_\infty, R_0]$ where a sequence of infinitely many bifurcations take place and study their compatibility with Feigenbaum's theory.

2. NUMERICAL RESULTS

Assuming the Lorenz system in the form

$$\begin{aligned} \dot{X} &= -\sigma X + \sigma Y \\ \dot{Y} &= -\sigma X - Y - XZ \qquad (\sigma = 10, b = 8/3) \\ \dot{Z} &= -bZ - R + XY \end{aligned} \tag{3}$$

obtained from the standard one (1) by changing the origin ($x \to X, y \to Y, z \to Z + \sigma + r$) and letting $R = b(\sigma + r)$, we can easily check numerically that for $R = 294$ any randomly chosen initial point tends to one of two symmetric periodic orbits.[2] Also it is easily verifiable that the Lorenz system retains the laminar behavior in a very narrow range of values of R: as a matter of fact, for two close values of R, for instance $R = 290$ and $R = 300$, we have clearly turbulent motion.

The computational techniques previously employed in Ref. 12, essentially based on Newton's method for searching periodic orbits and on

[2] The presence of such a pair of orbits is due to the symmetry $(X, Y, Z) \leftrightarrow (-X, -Y, Z)$ of the Lorenz equations. In the following we will consider only one of the two orbits, understanding that any statement holds unchanged for both.

Liapunov's criterion for examining their stability, allow us to study the two transitions from periodic to turbulent motion.

Denoting by Γ_0 the attractive periodic orbit found for $R = 294$ (Fig. 1), we get the following results.

(i) For increasing R, Γ_0 disappears for $R = R_0 = 295.453\ldots.$ An analysis of the eigenvalues of the Liapunov matrix relative to the Poincaré map shows that Γ_0 is always stable. An eigenvalue tends to 1 for R tending to R_0. One can easily verify that the disappearance of Γ_0 is due to the collapse to a hyperbolic orbit (Fig. 2).

(ii) For decreasing R, Γ_0 becomes unstable for $R = R_1 = 293.27\ldots$, since an eigenvalue of the Poincaré map crosses the unit circle at -1. As predicted by the bifurcation theory (see, for instance, Ref. 2), a new stable orbit Γ_1 (Fig. 3) appears, having a period which is twice that corresponding to Γ_0. The orbit Γ_1 becomes unstable at the bifurcation point $R_2 = 292.342\ldots$, where a new stable orbit Γ_2 appears having a period which is again doubled (Fig. 4). This sequence of orbit bifurcations continues and we find numerical evidence for Γ_3, Γ_4, and Γ_5 orbits by determining with sufficient accuracy the critical points $R_3 = 292.1256\ldots$, $R_4 = 292.07823\ldots$, and $R_5 = 292.06808\ldots.$

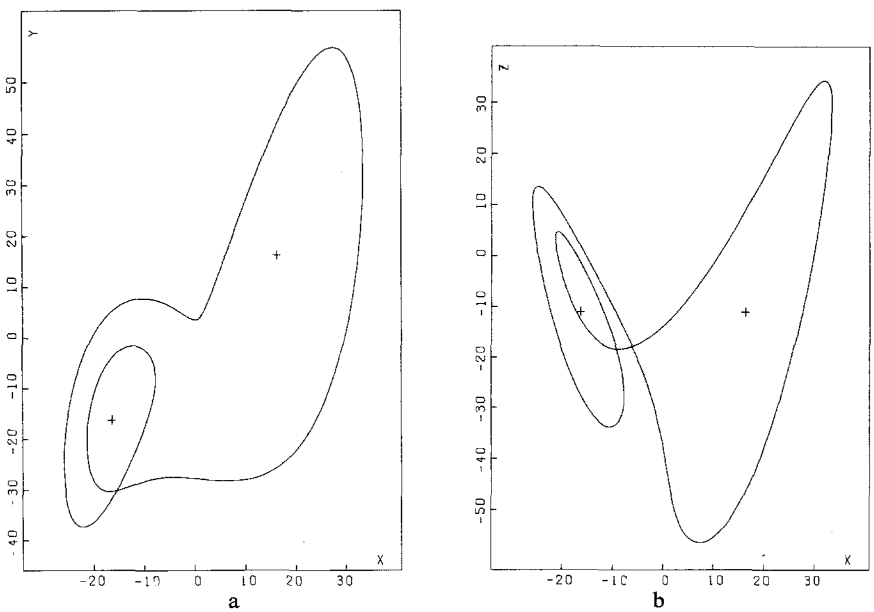

Fig. 1. Projections of the stable periodic orbit Γ_0 found for $R = 294$. Two crosses indicate the two fixed points $(\pm[-b(\sigma + 1) + R]^{1/2}, \pm[-b(\sigma + 1) + R]^{1/2}, -\sigma - 1)$.

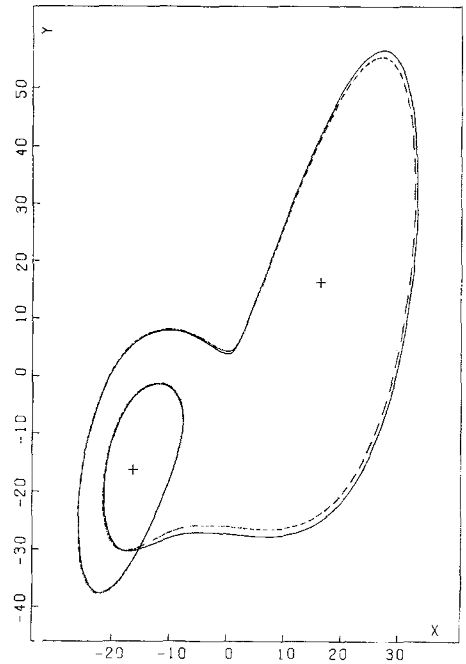

Fig. 2. Projection of the stable (—) and hyperbolic (---) orbits for $R = 295$. The two orbits appear to be very close to each other.

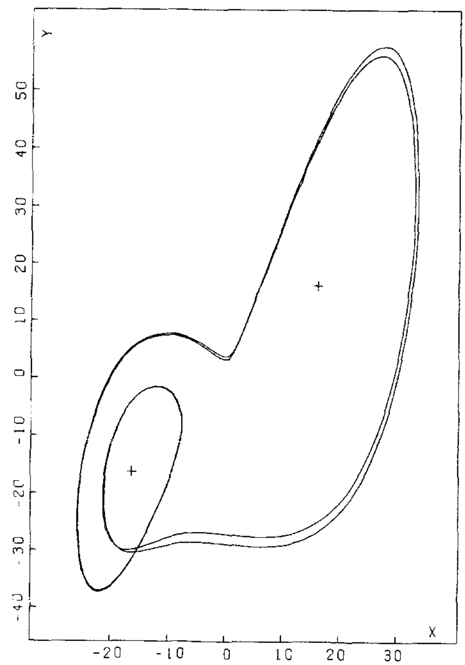

Fig. 3. Projection of the stable orbit Γ_1 for $R = 292.50$.

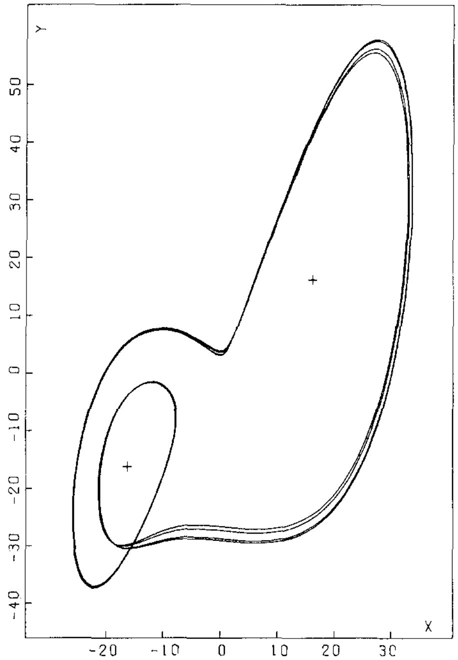

Fig. 4. Projection of the stable orbit Γ_2 for $R = 292.15$.

In Table I we collect most significant results, where R_0^+ is the minimum value of R that annihilates Γ_0; R_0^- is the maximum value of R that allows a stable Γ_0; T_0 is the period of Γ_0 for $R = R_0^-$; R_i^+, $i = 1, \ldots, 5$, is the minimum value of R that makes the orbit Γ_{i-1} stable, e.g., the eigenvalue of the Poincaré map that tends to cross the unit circle is still greater than -1; R_i^-, $i = 1, \ldots, 5$, is the maximum value of R that makes Γ_{i-1} already unstable, e.g., the eigenvalue is less than -1; T_i, $i = 1, \ldots, 5$, is the period of Γ_{i-1} for $R = R_i$.

Table I

i	R_i^+	R_i^-	T_i
0	295.454	295.453	1.094292
1	293.280	293.278	1.099559
2	292.3427	292.3426	2.203567
3	292.12565	292.12563	4.409210
4	292.078240	292.078230	8.819327
5	292.068087	292.068085	17.639042

Assuming $R_i = (R_i^+ + R_i^-)/2$ and computing the $\delta_i = (R_{i-1} - R_i)/(R_i - R_{i+1})$ ratios, $i = 1, \ldots, 4$, with estimated errors

$$\Delta\delta_i = \delta_i\left(\frac{\Delta R_{i-1} + \Delta R_i}{R_{i-1} - R_i} + \frac{\Delta R_i + \Delta R_{i+1}}{R_i - R_{i+1}}\right), \qquad \Delta R_i = \frac{R_i^+ - R_i^-}{2}$$

we obtain

$$\delta_1 = 2.322 \pm 0.005, \qquad \delta_2 = 4.315 \pm 0.007$$
$$\delta_3 = 4.578 \pm 0.003, \qquad \delta_4 = 4.671 \pm 0.005$$

which clearly agree with the limit value $\delta = 4.669\ldots$ as given by Feigenbaum.

It seems very likely indeed that the Γ_i sequence is infinite, so that one can tentatively estimate the critical value R_∞ in agreement with Feigenbaum's conjecture and with the numerical results reported above. Therefore one gets

$$R_\infty \approx R_5 + \frac{R_5 - R_4}{\delta} + \frac{R_5 - R_4}{\delta^2} + \ldots \approx 292.065320$$

For $R < R_\infty$ and $R > R_0$, the Lorenz system already exhibits turbulent behavior, as shown in Figs. 5 and 6, respectively, relative to $R = 292.00$ and

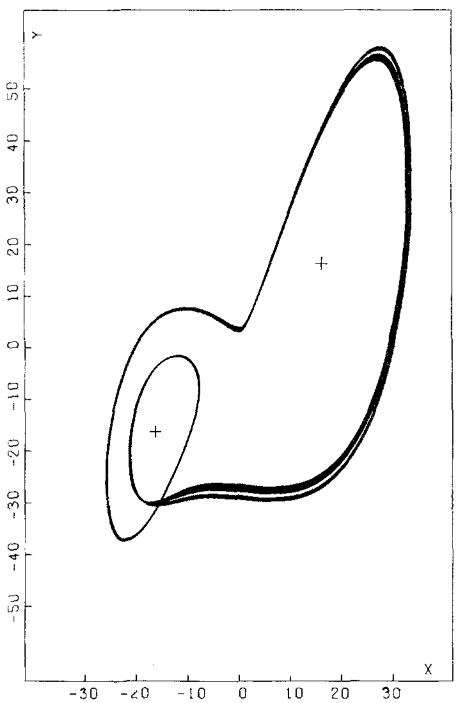

Fig. 5. Projection of the flow of a randomly chosen initial point within the time interval (100, 150) for $R = 292.00$. The motion takes place in a neighborhood of the infinite unstable orbits of one of the two sequences Γ_i.

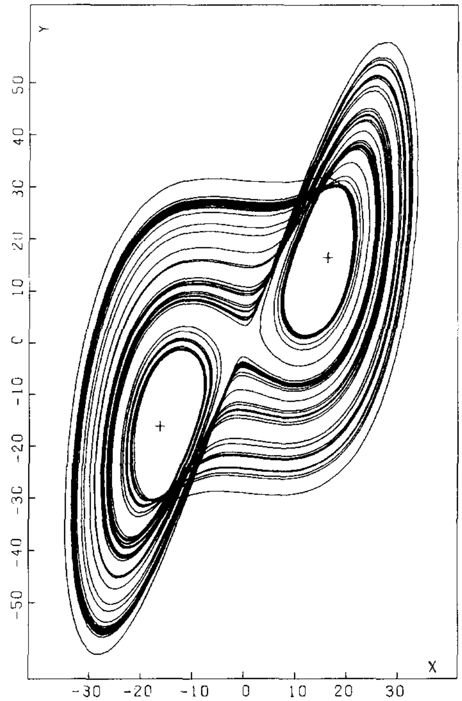

Fig. 6. Projection of the flow of a randomly chosen initial point within the time interval (100, 150) for $R = 295.50$. This picture clearly shows a strange attractor.

$R = 295.50$, e.g., to two values of R rather close to the two critical points. For $R = 292.00$ the motion could appear to be not completely chaotic: any randomly chosen initial point is attracted in a region surrounding the infinitely many unstable orbits of one of the two sequences Γ_i and therein trapped. The point can therefore move only in a neighborhood of these orbits, exactly as in Ref. 12 for the analogous case. An analysis carried out through the Poincaré map, following the same lines as in Ref. 12, which deals with two similar transitions from laminar to turbulent behavior, does not seem essential in the present case. As a matter of fact, it appears quite clear that these transitions occur with a phenomenology completely analogous to that already found through the system (2).

Finally, we briefly sketch the numerical methods employed to integrate the system (3). All the computed values reported above have been obtained by means of a fourth-order Runge-Kutta method (Butcher's method), imposing an error less than 10^{-5} in closing each orbit. Several computations have also been carried out by a third-order Runge-Kutta method. They provide results substantially equivalent to those reported above, the only difference being slightly translated bifurcation points (the differences $R_{i+1} - R_i$ were virtually invariant with respect to the integration method).

3. CONCLUSIONS

In the Lorenz model two important phenomena occur, which appear to be rather relevant, i.e., a sequence of infinite bifurcations and collapsing stable and hyperbolic orbits.

First, it is evident that these phenomena and the appearance (or disappearance) of turbulent behavior are strictly related. Taking into account the analysis reported in Ref. 12, one could reasonably infer that an attractive two-dimensional manifold also exists within the Lorenz model, with some singularities whose nature presumably changes as R varies. All phenomena shown by this model and investigated here could take place on that manifold.

Remarkably, the infinite sequence of bifurcations has been found to be compatible with Feigenbaum's theory. The fact that this theory is verified in the Lorenz model appears to be significant. It yields a second example in which the solutions of a first-order differential system have properties analogous to those of the map of an interval onto itself.

It is known that in the Lorenz model, as R keeps being increased starting from $R = b(\sigma + 470/19) \approx 92.63$, then, as Lanford states,[7] "the system appears to undergo a highly complicated sequence of bifurcations: for some R's there is an attracting periodic solution and for others a strange attractor." It seems therefore not unreasonable to assume the existence, for different ranges of the Rayleigh number, of further sequences of infinite periodic orbits, showing a phenomenology completely analogous to the present one.

In any event there are several points which probably need deeper analysis. Clearly, there is still some way to go and other numerical investigations appear necessary to throw light on the interesting aspects of this field.

Note. After the completion of this work, I was informed by P. Collet about a preprint by Robbins[16] in which another infinite sequence of bifurcations is found in the Lorenz model. Similar results have been found also by V. I. Yudovich (private communication via C. Boldrighini).

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